Let

$$f(x) = a_n x^n + \cdots + a_1 x + a_0.$$

The problem consists of determining kind of path what the local extrema of f trace out as we vary one coefficient a_i, leaving the other coefficients fixed. Assume that i is at least 1 (since we can easily see that tracing out a_0 gives a vertical line). I claim that the path is also the graph of a polynomial of degree n.

The local extrema occur at roots of $f'(x)$, where

$$f'(x) = na_n x^{n-1} + \cdots + ia_i x^{i-1} + \cdots + 2a_2 x + a_1.$$

Let x_0 be any of the $n-1$ such roots, that is, a solution to

$$na_n x_0^{n-1} + \cdots + ia_i x_0^{i-1} + \cdots + a_1 = 0.$$

In this equation every variable is fixed except a_i (the coefficient we are varying) and x_0 (the x-coordinate of one of the local extrema), so this equation gives a relationship between a_i and x_0. We are interested in the set of points $(x_0, f(x_0))$ as a_i varies over $\mathbb{R}$. To describe this we need to figure out what $f(x_0)$ is. We are almost able to do this, but we need to write a_i in terms of x_0. Fortunately the equation above is linear in a_i, so we have no problem solving for a_i:

$$a_i = -\frac{1}{ix_0^{i-1}}(na_n x_0^{n-1} + \cdots + (i+1)a_{i+1}x_0^i + (i-1)a_{i-1}x_0^{i-2} + \cdots + a_1).$$

Plug this into $f(x_0)$ (the first equation). The term $a_i x_0^i$ is the only messy term, equal to

$$-\frac{1}{ix_0^{i-1}}(na_n x_0^{n-1} + \cdots + (i+1)a_{i+1}x_0^i + (i-1)a_{i-1}x_0^{i-2} + \cdots + a_1)x_0^i$$

The $-\frac{1}{ix_0^{i-1}}$ term and x_0^i term cancel, leaving us with $-\frac{1}{i}x$ times a degree $n-1$ polynomial, which is of course a degree n polynomial in x_0. The rest of the terms are also a degree n polynomial, so the final result is a degree n polynomial in x_0. This is why the path traces out the graph of a degree n polynomial.