7.13.2015
Contest

Contest Paper - August Mock AMC 10

Instructions

1. Do not open this booklet until your have set your timer to 75 minutes.
2. FORMAT: This contest contains 25 problems ranging from relatively straighforward to extremely advanced. Problems are roughly in order of increasing difficulty (although there are definitely problems easier or harder than they should be). Each problem is followed by 5 answer choices labelled A, B, C, D, and E. Only one answer is correct. You will have 1 hour and 15 minutes to work on these 25 problems.
3. SCORING: You may only give 1 answer for each problem. If you give more than 1 answer, all answers will be marked as incorrect. Each correct answer is worth 6 points; each unanswered problem will earn 1.5 points.
4. Only scratch paper, graph paper, rulers, protractors, and compasses are allowed as an aid in this contest. Calculators, dictionaires, geometric objects, and other aids are not permitted.
5. Figures may not be drawn to scale.
6. Thanks for participating, and I hope you enjoy the contest!

Problem Contributions (AoPS Usernames)
Problems 1-11, 13-14, 16-18, 22-23 were proposed by azmath333.
Problems 12, 19, 20, 21, 24 were proposed by Lord.of.AMC.
Problem 15 was proposed by saagar.
Problem 25 was proposed by Tan.
Also, thank you to all contributors for making this contest possible!

1. Evaluate $\sqrt{1^3 + 2^3 + 3^3 \cdots 10^3}$.
(A) 4 **(B)** 10 **(C)** 27 **(D)** 55 **(E)** 120

2. Alice and Bob have 3 children. If each child is equally likely to be a boy or a girl, what is the probability that Alice and Bob have 2 boys and 1 girl?
(A) $\frac{1}{8}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{5}{8}$

3. I'm thinking of a positive integer larger than 1 that leaves a remainder of 1 when divided by each of 2, 3, 5, and 7. What is the smallest possible number I'm thinking of?
(A) 61 **(B)** 141 **(C)** 191 **(D)** 201 **(E)** 211

4. What is the product of the solutions of the equation $|x - 1| = 6$?
(A) -35 **(B)** -20 **(C)** 35 **(D)** 84 **(E)** 120

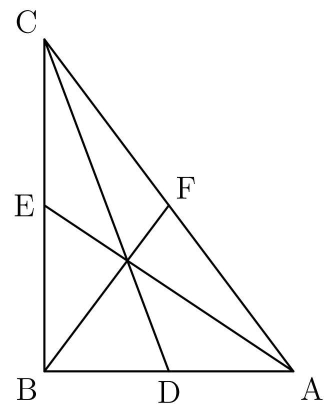
5. Define the function $f(x)$ as $f(x) = x + \frac{1}{x}$. What is the value of $(f(\sqrt{6}))^2$?
(A) 6 **(B)** $\frac{49}{6}$ **(C)** 8 **(D)** $\frac{77}{6}$ **(E)** $\frac{81}{6}$

6. A parallelogram has 3 consecutive vertices at (0,0), (1,5), and (4,5). What is the length of the shorter diagonal of this parallelogram?
(A) $\sqrt{13}$ **(B)** $\sqrt{15}$ **(C)** $\sqrt{23}$ **(D)** $\sqrt{29}$ **(E)** $\sqrt{39}$

7. Andrew, Bill, and Cleo are each asked to pick a number between 1 and 10, inclusive. What is the probability that the 3 of them picked distinct prime numbers?
(A) $\frac{3}{125}$ **(B)** $\frac{4}{125}$ **(C)** $\frac{1}{25}$ **(D)** $\frac{7}{125}$ **(E)** $\frac{63}{1000}$

8. The 4-digit number 19A0, where A represents a digit, is divisible by 11. What is the sum of all possible values of A?
(A) No solutions exist **(B)** 2 **(C)** 6 **(D)** 8 **(E)** 11

9. Triangle ABC has $AB = 6, BC = 8, CA = 10$. Let the midpoints of AB, BC, and CA be D, E, and F, respectively. Let the intersection point of medians DC, EA, and FB be P (not labeled). What is the area of ΔPDA?
(A) 2 **(B)** 2.5 **(C)** 3.2 **(D)** 3.75 **(E)** 4



10. How many 3-digit numbers have exactly 2 digits the same (while the third is different)?
(A) 81 **(B)** 162 **(C)** 243 **(D)** 252 **(E)** 261

11. The equation

$$xy - 4x + 2y = 8$$

is true when $x = a$ and/or when $y = b$. What is the value of $a + b$?
(A) -3 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 17

12. In country Z, the richest a% of the population share b% of the wealth. Assuming that the rest of the population shares the rest of the wealth equally, what *fraction* of the *total* wealth of country Z does each citizen not in the richest a% share?
(A) $\frac{100-a}{100-b}$ **(B)** $\frac{a}{b}$ **(C)** $\frac{100-b}{100(100-a)}$ **(D)** $\frac{100-b}{100-a}$ **(E)** $\frac{b}{100a}$

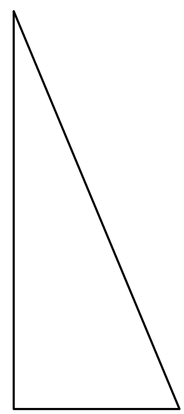
13. How many ordered pairs (x, y) satisfy the inequality $\frac{x^2+xy+y^2}{4} \geq \frac{3xy}{4}$?
(A) 0 **(B)** 1 **(C)** 2 **(D)** 27 **(E)** Infinitely many

14. A right isoceles triangle is inscribed in a circle with radius r. What is the area of this triangle, in terms of r?
(A) r^2 **(B)** $\sqrt{2}r^2$ **(C)** $2r^2$ **(D)** $2\sqrt{2}r^2$ **(E)** $3r^2$

15. A chicken wants to cross a road (don't ask why). The chicken, which is on one end of the road, is on a grid at the point $(0, 0)$. The other end of the road is $(5, 0)$. Every minute, the chicken randomly chooses to move up, down, right, or left 1 unit, and all four options are equally likely. What is the probability that after exactly 9 minutes, the chicken will be at $(5, 0)$? Express your answer as a common fraction.
(A) $\frac{27}{8192}$ **(B)** $\frac{81}{16384}$ **(C)** $\frac{243}{32768}$ **(D)** $\frac{729}{65536}$ **(E)** $\frac{2187}{131072}$

16. For what base k is the equation $202_k = 100_{10}$ true?
(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

17. Right triangle ΔXYZ has legs with length 5 and 12. Let the radius of the largest circle that can fit in the triangle be r, and let the radius of the circle that goes through all 3 vertices of the triangle be R. Find $r + 2R$.
(A) 6 **(B)** 7 **(C)** 13 **(D)** $\sqrt{191}$ **(E)** 15



18. In a swimming pool, there are 2 pumps and a drain. Pump A can fill the pool in 30 minutes, Pump B can fill the pool in 20 minutes, and the drain can empty the pool in 15 minutes. After filling the pool for 5 minutes only with Pump A (and with the drain closed) Pump B joined in. 5 minutes later, the drain was accidentally opened. After that, the pumps continue to fill the pool until it was full. How long (in minutes) was the entire process, starting from when Pump A started pumping and ending when the pool was full?
(A) 15 **(B)** 24 **(C)** 30 **(D)** 35 **(E)** 65

19. The numbers $3, 6, 9, \ldots, 300$ are given in a list. At each step of a process, you may take two distinct numbers from the list, find their positive difference, and replace the larger integer by this positive difference. This process ends when no more differences can be made. What is the smallest possible sum of the numbers in the list at any point in this process?
(A) 150 **(B)** 300 **(C)** 600 **(D)** 1200 **(E)** 15150

20. Let n be an integer so that $n^2 + 6n + 24$ is a perfect square. Suppose that m is the minimum possible value of n, and M is the largest possible value. What is $M + m$?
(A) -6 **(B)** -3 **(C)** 0 **(D)** 6 **(E)** 16

21. Fifty-eight candies are distributed among five boxes. Let a_n be the number of candies in the n-th box, starting from 1. It is known that $3 \leq a_1 \leq 8$, $6 \leq a_2 \leq 8$, $9 \leq a_3 \leq 12$, $12 \leq a_4 \leq 20$, and $a_5 = 15$. Aniwey is allowed to open two of the five candy boxes. Assuming he chooses optimally, what is the maximum number of candies that he is guaranteed to obtain?
(A) 27 **(B)** 28 **(C)** 29 **(D)** 30 **(E)** 31

22. Given the system of equations

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{23}{28},$$
$$\frac{1}{xy} + \frac{1}{yz} + \frac{1}{xz} = \frac{5}{28},$$
$$\frac{3}{xyz} = \frac{3}{112},$$

find $\frac{x^2yz+xy^2z+xyz^2}{z}$ if $x > y > z$.
(A) $\frac{1}{112}$ **(B)** $\frac{1}{2}$ **(C)** 28 **(D)** 280 **(E)** 1120

23. AoPS-City has a new row of 5 blocks marked for new development. A house, apartment, and hotel each take up 1 block of land, while a mansion and a school each take up 2 blocks. How many ways are there to allocate these 5 blocks if these 5 buildings are the only options?
(A) 441 **(B)** 484 **(C)** 495 **(D)** 560 **(E)** 640

24. Define the digits in base 60 as $d_0, d_1, d_2, \cdots, d_{59}$, where d_i represents the base-10 numeral i. Define each term of a sequence $(a_1, \cdots, a_{59})$ as $a_i = d_1 d_2 \cdots d_{i-1} d_i d_{i-1} d_{i-2} \cdots d_2 d_1$ (i.e., a_i consists of $2i - 1$ base-60 digits). For example, $a_3 = d_1 d_2 d_3 d_2 d_1$. Suppose that, in base-60 division, a_{58}/a_{29} is some sequence of digits in base 60, and let the base-10 equivalents of those digits be $n_1, n_2, \cdots, n_k$. Determine $n_1 + n_2 + \cdots + n_k$.
(A) 4 **(B)** 40 **(C)** 400 **(D)** 4000 **(E)** 40000

25. Suppose I is the incenter of ΔABC. The line AI meets BC and the circumcircle of ΔABC at D and E respectively. Suppose $\angle BAE = 45°, BD = \frac{60}{7}, CD = \frac{45}{7}$ and $DE = \frac{75\sqrt{2}}{14}$. The length ID can be expressed as $\frac{a\sqrt{b}}{c}$. Compute the value of $a + b + c$.
(A) 24 **(B)** 42 **(C)** 68 **(D)** 84 **(E)** 120